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On the Theorem of Feit-Higman

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We let correspond to any finite incidence structure S a certain semisimple algebra of endomorphisms of the vector space spanned by the flags of S. In case the structure is a generalized polygon we compute the irreducible representations of the algebra and deduce the theorem of Feit–Higman.

1. INTRODUCTION

A generalized n-gon is an incidence structure S consisting of a set P of points, a set L of lines, and certain axioms of incidence. In 1964 Feit and Higman [5] proved the following:

THEOREM. *Let S be a finite generalized n-gon with $s + 1$ points on every line and $t + 1$ lines on every point. Then either $s = 1 = t$ and S is an ordinary n-gon, or $n \in \{2, 3, 4, 6, 8, 12\}$. If $s > 1$ and $t > 1$ then $n \in \{2, 3, 4, 6, 8\}$, and st is a square if $n = 6$, $2st$ is a square if $n = 8$.*

In this paper we give a proof which is similar to theirs in spirit but simpler in structure. Let V be the vector space over the complex field $\mathbf{C}$ which has the set of flags of S as basis. Since S is finite, V has finite dimension. We associate with S a semisimple $\mathbf{C}$-algebra A of endomorphisms of V. Since V is an A-module it affords a character ϕ_V of A. We may write

$$\phi_V = \sum n_\psi \psi$$

where the ψ are distinct irreducible characters of A and the multiplicities n_ψ are non-negative integers. The space of complex valued functions on A carries a non-degenerate symmetric bilinear form (,) under which distinct irreducible characters are orthogonal. Thus

$$(\phi_V, \psi) = n_\psi(\psi, \psi).$$

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One sees easily that $(\phi_V, \psi) = \psi(1)$ is a positive integer. Thus (ψ, ψ) is rational. On the other hand (ψ, ψ) may be computed directly from explicit formulas for the irreducible characters and its rationality forces the assertion of the theorem.

We use Schur's method to prove the orthogonality of the characters. The same idea may be applied to a class of algebras which includes group algebras, centralizer algebras of permutation groups, and the algebras defined by combinatorial association schemes [4, §7.1]. In the case of centralizer algebras the orthogonality relations are those of Curtis and Fossum [1], and Tamaschke [8].

Although the argument assumes nothing about finite groups with (B, N)-pair it has been very much influenced by the papers of Iwahori [6, 7] on Hecke algebras, and by a paper of Curtis, Iwahori, and Kilmoyer on representations of finite groups with (B, N)-pair [2]. The second author is happy to acknowledge his debt to Iwahori for several seminars in the Japanese style.

2. Definition and Structure of the Algebra

By an incidence structure, we mean a triple (P, L, F) where P, L are sets and F is a subset of $P \times L$. Elements $p \in P$ are called points, elements $l \in L$ are called lines, and pairs $(p, l) \in F$ are called flags. If $(p, l) \in F$ we say that p is on l or that l is on p, or that p and l are incident. We assume throughout that there are $s + 1$ points on every line and $t + 1$ lines on every point, for some positive integers s, t and that P and L are finite. If $x = (p, l)$ and $y = (q, m)$ are distinct flags we write $x \cap y = p$ if $p = q$ and write $x \cap y = l$ if $l = m$. If $x \in F$, define subsets $S(x)$, $T(x)$ of F by

$$S(x) = \{y \in F \mid x \cap y \in L\}, \qquad T(x) = \{y \in F \mid x \cap y \in P\}.$$

Let V be the $\mathbf{C}$-space with F as basis. Since P and L are finite we may define endomorphisms σ, τ of V by

$$\sigma x = \sum_{y \in S(x)} y, \qquad \tau x = \sum_{y \in T(x)} y. \tag{1}$$

Let A be the algebra of endomorphisms generated by σ and τ.

Lemma 1. *The endomorphisms σ, τ satisfy the relations*

$$\sigma^2 = s \cdot 1 + (s - 1)\sigma, \qquad \tau^2 = t \cdot 1 + (t - 1)\tau. \tag{2}$$

Proof. Let $x = (p, l) \in F$. Let $p, p_1, \ldots, p_s$ be the points on l. Then $S(x)$ consists of the flags (p_i, l) for $i = 1, \ldots, s$. Choose some flag $y = (p_i, l)$. Then $S(y)$ consists of the flags (p, l) and (p_j, l) with $j \neq i$.

Thus

$$\sigma^2 x = \sum_{i=1}^{s} \Big[(p, l) + \sum_{j \neq i} (p_j, l) \Big] = sx + (s - 1)\, \sigma x.$$

The argument for τ is similar.

The notion of generalized n-gon is due to Tits. The definition, and brief description of the known n-gons appears in [4, §7.3]. We repeat the definition. Let (P, L, F) be an incidence structure. If $a, b \in P \cup L$, a chain of length h from a to b is a sequence $a = a_0, a_1, \ldots, a_h = b$ of elements of $P \cup L$ such that a_{i-1} and a_i are incident for all $i = 1, \ldots, h$. Note that if $a_{i-1} \in P$ then $a_i \in L$. We define $\rho(a, b)$ to be the least integer h (if any exists) such that there exists a chain of length h from a to b. A generalized n-gon is an incidence structure such that

(i) $\rho(a, b) \leqslant n$ for all $a, b \in P \cup L$.

(ii) If $\rho(a, b) = h < n$ then there is only one chain of length h from a to b.

(iii) Given $a \in P \cup L$ there exists $b \in P \cup L$ such that $\rho(a, b) = n$.

Note that the set P of vertices and the set L of edges of an ordinary n-gon, together with the natural incidences, satisfy (i)–(iii) with $s = 1 = t$. We assume henceforth that (P, L, F) is a generalized n-gon. Our first task is to determine the structure of the algebra A.

If $x, y \in F$ we say that x and y are adjacent if $x \cap y$ is defined and either $x \cap y \in P$ or $x \cap y \in L$. A gallery [9, §1] of length k from x to y is a sequence $x = x_0, x_1, \ldots, x_k = y$ of flags such that x_{i-1} and x_i are adjacent for all $i = 1, \ldots, k$. If $x, y \in F$ we define $\operatorname{dist}(x, y)$ to be the least integer k such that there exists a gallery of length k from x to y. A gallery of length k from x to y is minimal if $k = \operatorname{dist}(x, y)$. Note that in any minimal gallery, $x_{i-1} \cap x_i \in P$ implies $x_i \cap x_{i+1} \in L$. The following Lemmas 2–5 are easy consequences of the axioms. The reader is encouraged to draw a graph in which $P \cup L$ is the set of vertices and there is an edge for each incident point-line pair. One should use Lemma 2 to prove Lemmas 3 and 4. Lemma 5 appears in [5].

LEMMA 2. *Let (P, L, F) be a generalized n-gon and let $x, y \in F$. Then*

(i) $\operatorname{dist}(x, y) \leqslant n$.

(ii) *If* $\operatorname{dist}(x, y) < n$ *there exists a unique minimal gallery from x to y.*

(iii) *If* $\operatorname{dist}(x, y) = n$ *there are precisely two minimal galleries $x = x_0, x_1, \ldots, x_n = y$. One of these has the property $x_{n-1} \cap y \in P$ and the other has the property $x_{n-1} \cap y \in L$.*

For each $x \in F$ and positive integer k, define subsets $S_k(x)$, $T_k(x)$ of F as follows. Say $y \in S_k(x)$ if there exists a minimal gallery $x = x_0, \ldots, x_k = y$ from x to y such that $x_{k-1} \cap y \in L$. Say $y \in T_k(x)$ if there exists a minimal gallery $x = x_0, \ldots, x_k = y$ from x to y such that $x_{k-1} \cap y \in P$. Thus $S_1(x) = S(x)$ and $T_1(x) = T(x)$. We agree that $S_0(x) = \{x\} = T_0(x)$.

LEMMA 3. *$S_k(x) \cap T_k(x)$ is empty for $k = 1, \ldots, n-1$ and $S_n(x) = T_n(x)$.*

LEMMA 4. *Suppose $1 \leqslant k \leqslant n$ and $x, y \in F$. Then*

(i) *$y \in S_k(x)$ if and only if there exists a unique $z \in T_{k-1}(x)$ such that $y \in S(z)$.*

(ii) *$y \in T_k(x)$ if and only if there exists a unique $z \in S_{k-1}(x)$ such that $y \in T(z)$.*

LEMMA 5. *If n is odd then $s = t$.*

THEOREM 1. *Let (P, L, F) be a finite generalized n-gon. Let A be the algebra of endomorphisms generated by σ and τ. Then* $\dim A = 2n$ *and A had defining relations*

$$\sigma^2 = s \cdot 1 + (s-1)\sigma, \qquad \tau^2 = t \cdot 1 + (t-1)\tau; \tag{2}$$

$$\begin{aligned} (\sigma\tau)^m &= (\tau\sigma)^m, && \text{if } n = 2m \text{ is even,} \\ (\sigma\tau)^m \sigma &= (\tau\sigma)^m \tau, && \text{if } n = 2m+1 \text{ is odd.} \end{aligned} \tag{3}$$

Proof. Define endomorphisms σ_k, τ_k of V by

$$\sigma_k x = \sum_{y \in S_k(x)} y, \qquad \tau_k x = \sum_{y \in T_k(x)} y. \tag{4}$$

It follows from Lemma 3 that

$$\sigma\tau_{k-1} x = \sum_{z \in T_{k-1}(x)} \sigma z = \sum_{z \in T_{k-1}(x)} \sum_{y \in S(z)} y = \sum_{y \in S_k(x)} y = \sigma_k x$$

so that $\sigma\tau_{k-1} = \sigma_k$. Similarly $\tau\sigma_{k-1} = \tau_k$. Since $\sigma_1 = \sigma$ and $\tau_1 = \tau$ we conclude that

$$\sigma_{2k-1} = (\sigma\tau)^{k-1}\sigma, \quad \sigma_{2k} = (\sigma\tau)^k, \quad \tau_{2k-1} = (\tau\sigma)^{k-1}\tau, \quad \tau_{2k} = (\tau\sigma)^k. \tag{5}$$

From Lemma 3 we have $S_n(x) = T_n(x)$ so that $\sigma_n = \tau_n$ and thus the relation (3) is satisfied. From (2) and (3) we conclude that $\dim A \leqslant 2n$. On the other hand, since the sets $S_i(x) \cap S_j(x)$, $T_i(x) \cap T_j(x)$, and

$S_i(x) \cap T_j(x)$ are certainly empty for $i \neq j$ it follows from Lemma 3 that F is a disjoint union of the sets

$$S_0(x) = T_0(x), S_1(x),..., S_{n-1}(x), T_1(x),..., T_{n-1}(x), S_n(x) = T_n(x).$$

Thus

$$\Gamma = \{\sigma_0 = \tau_0, \sigma_1, ..., \sigma_{n-1}, \tau_1, ..., \tau_{n-1}, \sigma_n = \tau_n\} \tag{6}$$

is a linearly independent set of endomorphisms in A so in fact $\dim A = 2n$ and Γ is a basis for A. If A' is the algebra with identity defined by generators σ', τ' and relations (2), (3) with σ, τ replaced by σ', τ' then we must have $\dim A' \leqslant 2n$. Since A is a homomorphic image of A' and $\dim A = 2n$ we conclude that $A \simeq A'$.

LEMMA 6. *A has an antiautomorphism which fixes both σ and τ.*

Proof. Let X be the free associative $\mathbf{C}$-algebra on two generators ξ, η. The $\mathbf{C}$-linear map θ of X into X defined by $\theta 1 = 1$, $\theta(\zeta_1 \cdots \zeta_p) = \zeta_p \cdots \zeta_1$ for all $\zeta_i \in \{\xi, \eta\}$, is an antiautomorphism of X which fixes ξ, η. Let I be the ideal of X generated by the elements $\xi^2 - s \cdot 1 - (s-1)\xi$, $\eta^2 - t \cdot 1 - (t-1)\eta$, and $(\xi\eta)^m - (\eta\xi)^m$ if $n = 2m$ is even, $(\xi\eta)^m\xi - (\eta\xi)^m\eta$ if $n = 2m + 1$ is odd. Since $\theta(I) \subseteq I$ we get an induced antiautomorphism in the quotient $X/I \simeq A$.

3. REPRESENTATIONS OF THE ALGEBRA A

If M is an A-module and $\alpha \in A$ we let α_M denote the endomorphism defined by $\alpha_M x = \alpha x$, $x \in M$. Recall from the proof of Theorem 1 that A has a distinguished basis $\Gamma = \{\sigma_0, ..., \sigma_{n-1}, \tau_1, ..., \tau_n\}$.

LEMMA 7. *There exists an algebra homomorphism from A to $\mathbf{C}$ which maps σ to s and τ to t.*

Proof. Since $s^2 = s + (s-1)s$, $t^2 = t + (t-1)t$, $(st)^m = (ts)^m$ if $n = 2m$, and $(st)^m s = (ts)^m t$ if $n = 2m + 1$ because $s = t$ when n is odd, the assertion follows from Theorem 1.

Let ind: $A \to \mathbf{C}$ denote the homomorphism of Lemma 7. Note that ind does not vanish on the elements of Γ. Let $\alpha \to \alpha'$, $\alpha \in A$, denote the antiautomorphism of Lemma 6.

LEMMA 8. *Let M, N be A-modules and suppose $\xi \in \mathrm{Hom}_{\mathbf{C}}(M, N)$. Let $\zeta = \sum_{\gamma \in \Gamma} (\mathrm{ind}\ \gamma)^{-1} \gamma_N \xi \gamma_M'$. Then $\xi \in \mathrm{Hom}_A(M, N)$.*

Proof. Since σ and τ generate A, it suffices to prove $\zeta\sigma_M = \sigma_N\zeta$ and $\zeta\tau_M = \tau_N\zeta$. We prove the assertion for σ; the argument for τ is the same. Define integers $c(\gamma, \beta)$ for $\gamma, \beta \in \Gamma$ by

$$\sigma\gamma = \sum_{\beta\in\Gamma} c(\gamma, \beta)\beta. \tag{7}$$

Since $\sigma\sigma_i = s\tau_{i-1} + (s-1)\sigma_i$ and $\sigma\tau_{i-1} = \sigma_i$ for all $i = 1,\dots, n$ the integers $c(\gamma, \beta)$ are given by

$$\begin{aligned}
c(\sigma_i, \tau_{i-1}) &= s, & c(\sigma_i, \tau_{j-1}) &= 0, \quad \text{if } j \neq i,\\
c(\sigma_i, \sigma_i) &= s-1, & c(\sigma_i, \sigma_j) &= 0, \quad \text{if } j \neq i,\\
c(\tau_{i-1}, \sigma_i) &= 1, & c(\tau_{i-1}, \sigma_j) &= 0, \quad \text{if } j \neq i,\\
c(\tau_{i-1}, \tau_{j-1}) &= 0, & &\text{for all } i, j,
\end{aligned}$$

where i, j range over $1,\dots, n$. From these formulas we see that

$$c(\gamma, \beta)(\operatorname{ind}\gamma)^{-1} = c(\beta, \gamma)(\operatorname{ind}\beta)^{-1} \tag{8}$$

for all $\beta, \gamma \in \Gamma$. Since $'$ is an antiautomorphism which fixes σ, (7) implies

$$\gamma'\sigma = \sum_{\beta\in\Gamma} c(\gamma, \beta)\beta'. \tag{9}$$

Now (7), (8), and (9) imply

$$\begin{aligned}
\sigma_N\zeta &= \sum_{\gamma\in\Gamma}\sum_{\beta\in\Gamma} (\operatorname{ind}\gamma)^{-1} c(\gamma, \beta)\, \beta_N\xi\gamma_M'\\
&= \sum_{\beta\in\Gamma}\sum_{\gamma\in\Gamma} (\operatorname{ind}\beta)^{-1} c(\beta, \gamma)\, \gamma_N\xi\beta_M'\\
&= \sum_{\beta\in\Gamma}\sum_{\gamma\in\Gamma} (\operatorname{ind}\gamma)^{-1} c(\gamma, \beta)\, \gamma_N\xi\beta_M' = \zeta\sigma_M,
\end{aligned}$$

which proves the assertion.

If ϕ, ψ are complex valued functions on A we define

$$(\phi, \psi) = |F|^{-1} \sum_{\gamma\in\Gamma} (\operatorname{ind}\gamma)^{-1} \phi(\gamma)\, \psi(\gamma').$$

If M is an A-module, the character afforded by M is the complex valued function on A defined by $\phi_M(\alpha) = \operatorname{trace} \alpha_M$.

LEMMA 9. *Let M, N be simple nonisomorphic A-modules. Then* $(\phi_M, \phi_N) = 0$.

Proof. It follows from Lemma 7 and Schur's lemma that

$$\sum_{\gamma \in \Gamma} (\operatorname{ind} \gamma)^{-1} \gamma_N \xi \gamma_M = 0$$

for all $\xi \in \operatorname{Hom}_{\mathbf{C}}(M, N)$. Now Schur's argument [3, §31] for the orthogonality of characters of finite groups yields the result.

Unlike the situation with group characters, one need not have $(\phi_M, \phi_M) = 1$ for a simple module M. To compute (ϕ_M, ϕ_M) we need an explicit formula for the character. We know from an argument of Iwahori and Tits [7] that A is isomorphic to the complex group algebra $\mathbf{C}[W]$ of a dihedral group W of order $2n$. Although this fact is of no help to us in the calculations since we know no explicit formula for the isomorphism, it does suggest the following construction of the representations of A, which degenerates into one for $\mathbf{C}[W]$ in case $s = 1 = t$. If a, b are any complex numbers let

$$S = S(a) = \begin{pmatrix} -1 & a \\ 0 & s \end{pmatrix}, \qquad T = T(b) = \begin{pmatrix} t & 0 \\ b & -1 \end{pmatrix}. \tag{10}$$

Note that

$$S^2 = sI + (s-1)S, \qquad T^2 = tI + (t-1)T \tag{11}$$

for all choices of a and b, where I is the identity matrix. In view of Theorem 1, to produce a two-dimensional representation of A we must simply choose a, b so that the relations (3) are satisfied with S, T in place of σ, τ.

THEOREM 2. *Let (P, L, F) be a generalized n-gon with $s + 1$ points on each line and $t + 1$ lines on each point. Let $s_0 = \sqrt{s}$, $t_0 = \sqrt{t}$, $\theta_j = 2\pi j/n$ and a_j, b_j any complex numbers such that $a_j b_j = s + t + 2 s_0 t_0 \cos \theta_j$. If $n = 2m$ is even then A has four representations* $\operatorname{ind} = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ *of degree* 1 *given by*

$$\lambda_1(\sigma) = s, \quad \lambda_1(\tau) = t; \qquad \lambda_2(\sigma) = -1, \quad \lambda_2(\tau) = t;$$
$$\lambda_3(\sigma) = s, \quad \lambda_3(\tau) = -1; \qquad \lambda_4(\sigma) = -1, \quad \lambda_4(\tau) = -1$$

and $m - 1$ inequivalent irreducible representations $F_1, \ldots, F_{m-1}$ of degree 2 *given by*

$$F_j(\sigma) = S(a_j), \qquad F_j(\tau) = T(b_j).$$

If $n = 2m + 1$ is odd then A has two representations $\operatorname{ind} = \lambda_1, \lambda_2$ *of degree* 1 *given by*

$$\lambda_1(\sigma) = s, \quad \lambda_1(\tau) = t; \qquad \lambda_2(\sigma) = -1, \quad \lambda_2(\tau) = -1$$

and m inequivalent irreducible representations $F_1, \ldots, F_m$ of degree 2 given by

$$F_j(\sigma) = S(a_j), \qquad F_j(\tau) = T(b_j).$$

These are, up to equivalence, all the irreducible representations of A. Moreover if ϕ_j is the character of F_j then

$$\phi_j(\sigma_{2k}) = \phi_j(\tau_{2k}) = 2(s_0t_0)^k \cos k\theta_j \tag{12}$$

$$\phi_j(\sigma_{2k+1}) = (s_0t_0)^{k-1} (\sin \theta_j)^{-1} [s(t-1) \sin k\theta_j + s_0t_0(s-1) \sin(k+1)\, \theta_j] \tag{13}$$

$$\phi_j(\tau_{2k+1}) = (s_0t_0)^{k-1} (\sin \theta_j)^{-1} [t(s-1) \sin k\theta_j + s_0t_0(t-1) \sin(k+1)\, \theta_j]. \tag{14}$$

Proof. There is nothing to be said about the representations λ of degree 1 since the relations (2), (3) are satisfied with $\lambda(\sigma)$ and $\lambda(\tau)$ in place of σ and τ. Let $\theta_j = \theta$, $a_j = a$, $b_j = b$, $S = S(a)$, $T = T(b)$. Then

$$ST = \begin{pmatrix} ab - t & -a \\ bs & -s \end{pmatrix}, \qquad TS = \begin{pmatrix} -t & at \\ -b & ab - s \end{pmatrix}.$$

It follows from our assumption $ab = s + t + 2s_0t_0 \cos \theta$ that ST and TS have the eigenvalues $s_0t_0e^{\pm i\theta}$. Let P be a matrix such that

$$P^{-1}STP = \begin{pmatrix} s_0t_0e^{i\theta} & 0 \\ 0 & s_0t_0e^{-i\theta} \end{pmatrix} \tag{15}$$

and put $S' = P^{-1}SP$, $T' = P^{-1}TP$. Let $D = P^{-1}STP$ and let

$$S' = \begin{pmatrix} x & u \\ v & y \end{pmatrix}. \tag{16}$$

Then

$$T' = S'^{-1}D = \begin{pmatrix} -s^{-1}s_0t_0e^{i\theta}y & s^{-1}s_0t_0e^{-i\theta}u \\ s^{-1}s_0t_0e^{i\theta}v & -s^{-1}s_0t_0e^{-i\theta}x \end{pmatrix}. \tag{17}$$

From the fact that trace $S' = s - 1$, trace $T' = t - 1$ we obtain

$$x = (2is_0t_0 \sin \theta)^{-1} [s(t-1) + s_0t_0(s-1)\, e^{i\theta}], \tag{18}$$

$$y = -(2is_0t_0 \sin \theta)^{-1} [s(t-1) + s_0t_0(s-1)\, e^{-i\theta}]. \tag{19}$$

Now, if $n = 2m$ is even, then $e^{2mi\theta} = 1$ and hence

$$(ST)^m = (TS)^m = D^m = \pm(s_0t_0)^m \cdot I.$$

On the other hand, if $n = 2m + 1$ is odd, then $s = t$ and it follows from (18), (19) that $y = -e^{-i\theta}x$. Thus from (17)

$$T' = \begin{pmatrix} x & ue^{-i\theta} \\ ve^{i\theta} & y \end{pmatrix}.$$

But then since $e^{(2m+1)i\theta} = 1$ we have $D^m S' = T'D^m$, $(S'T')^m S' = T'(S'T')^m$, and hence $(ST)^m S = T(ST)^m$. Thus in either case the relations (2), (3) are satisfied with S and T in place of σ and τ showing that the representations F_j may be defined as in the statement of the theorem. Let ϕ_j be the character afforded by F_j. Since $\phi_j(\sigma\tau) = 2s_0t_0 \cos \theta_j$, distinct j give rise to distinct ϕ_j and hence to inequivalent representations F_j. Since the only matrices which commute with both $F_j(\sigma)$ and $F_j(\tau)$ are scalar multiples of the identity matrix, F_j is irreducible. The sum of the squares of the degrees of the representations we have constructed is $4 \cdot 1^2 + (m-1)\, 2^2 = 4m = 2n$ if n is even and is $2 \cdot 1^2 + m \cdot 2^2 = 2(2m+1) = 2n$ if n is odd. Since $\dim A = 2n$ we see that A is semisimple and we have constructed all the irreducible representations. Finally let $F_j = F$, $\theta_j = \theta$, and $\phi_j = \phi$ be the character afforded by F. Since ST and TS are both similar to D we have

$$\phi(\sigma_{2k}) = \phi(\tau_{2k}) = \operatorname{trace} D^k = 2(s_0t_0)^k \cos k\theta.$$

From (16), (18), (19) we have

$$\begin{aligned}\phi(\sigma_{2k+1}) &= \operatorname{trace}(ST)^k S = \operatorname{trace} D^k S' \\ &= (s_0t_0)^k (e^{ik\theta}x + e^{-ik\theta}y) \\ &= (s_0t_0)^{k-1} (\sin\theta)^{-1} [s(t-1)\sin k\theta + (s_0t_0)(s-1)\sin(k+1)\theta].\end{aligned}$$

Similarly

$$\begin{aligned}\phi(\tau_{2k+1}) &= \operatorname{trace} T(ST)^k = \operatorname{trace} T'D^k \\ &= (s_0t_0)^{k-1} (\sin\theta)^{-1} [t(s-1)\sin k\theta + s_0t_0(t-1)\sin(k+1)\theta].\end{aligned}$$

LEMMA 10. *If $n = 2m$ let $\theta_j = j\pi/m$. Then*

$$|F|(\phi_j, \phi_j) = 4m + \left[\frac{(s-1)^2}{s} + \frac{(t-1)^2}{t}\right] \frac{m}{\sin^2\theta_j} + \frac{(s-1)(t-1)}{s_0t_0} \frac{2m\cos\theta_j}{\sin^2\theta_j}. \tag{20}$$

If $n = 2m + 1$ let $\theta_j = 2j\pi/(2m+1)$. Then

$$|F|(\phi_j, \phi_j) = 4m + 2 + \frac{(s-1)^2}{s} \frac{2m+1}{1-\cos\theta_j}. \tag{21}$$

Proof. Let $\theta = \theta_j$, $\phi = \phi_j$ and if $\gamma \in \Gamma$ let $f(\gamma) = (\operatorname{ind} \gamma)^{-1} \phi(\gamma)^2$. From (12), (13), and (14) we conclude

$$f(\sigma_{2k}) + f(\tau_{2k}) = 8 \cos^2 k\theta, \tag{22}$$

$$\begin{aligned} f(\sigma_{2k+1}) &+ f(\tau_{2k+1}) \\ &= \left[\frac{(s-1)^2}{s} + \frac{(t-1)^2}{t}\right] \frac{1}{\sin^2 \theta} (\sin^2 k\theta + \sin^2(k+1)\theta) \\ &\quad + \frac{4(s-1)(t-1)}{s_0 t_0} \frac{1}{\sin^2 \theta} \cdot \sin k\theta \sin(k+1)\theta. \end{aligned} \tag{23}$$

Suppose $n = 2m$. Then $\cos m\theta = \pm 1$ and (12) implies $f(\sigma_0) = 4 = f(\sigma_{2m})$. Thus

$$\sum_{\gamma \in \Gamma} f(\gamma) = \sum_{k=0}^{m-1} [f(\sigma_{2k}) + f(\tau_{2k}) + f(\sigma_{2k+1}) + f(\tau_{2k+1})]$$

$$= 8 \sum_{k=0}^{m-1} \cos^2 k\theta + \left[\frac{(s-1)^2}{s} + \frac{(t-1)^2}{t}\right] \frac{1}{\sin^2 \theta} \sum_{k=0}^{m-1} \sin^2 k\theta + \sin^2(k+1)\theta$$

$$+ \frac{4(s-1)(t-1)}{s_0 t_0} \frac{1}{\sin^2 \theta} \sum_{k=0}^{m-1} \sin k\theta \sin(k+1)\theta.$$

Computing the trigonometric sums is tedious, but amounts simply to summing geometric series if we write the trigonometric functions in exponential form. Using $e^{2mi\theta} = 1$ we arrive at (20). Now suppose $n = 2m + 1$. Since $s = s_0 t_0 = t$ we may rewrite (23) as

$$\begin{aligned} f(\sigma_{2k+1}) + f(\tau_{2k+1}) &= \frac{2(s-1)^2}{s} \left[\frac{\sin k\theta + \sin(k+1)\theta}{\sin \theta}\right]^2 \\ &= \frac{2(s-1)^2}{s} \frac{\sin^2(k+\frac{1}{2})\theta}{\sin^2 \frac{1}{2}\theta}. \end{aligned}$$

Now (13) implies $f(\sigma_{2m+1}) = 0$ so that

$$\begin{aligned} \sum_{\gamma \in \Gamma} f(\gamma) &= -4 + \sum_{k=0}^{m} [f(\sigma_{2k}) + f(\tau_{2k}) + f(\sigma_{2k+1}) + f(\tau_{2k+1})] \\ &= -4 + 8 \sum_{k=0}^{m} \cos^2 k\theta + \frac{2(s-1)^2}{s} \frac{1}{\sin^2 \frac{1}{2}\theta} \sum_{k=0}^{m-1} \sin^2(k+\tfrac{1}{2})\theta. \end{aligned}$$

In this case the trigonometric sums are computed using $e^{2(m+1)i\theta} = 1$ and the result is (21).

4. The Theorem of Feit-Higman

Let ϕ_V be the character of A afforded by the representation of A on V and write $\phi_V = \sum n_\psi \psi$ where the sum is over all irreducible characters ψ of A. The multiplicities n_ψ are non-negative integers. If $\gamma \in \Gamma$ and $\gamma \neq 1$ then the defining formulas (4) show for $x \in F$ that γx is a sum of flags distinct from x and thus $\phi_V(\gamma) = 0$. On the other hand $\phi_V(1) = |F|$. Thus $(\phi_V, \psi) = \psi(1)$. We conclude from the orthogonality relations that $\psi(1) = n_\psi(\psi, \psi)$ so that $(\psi, \psi) \in \mathbf{Q}$ for every irreducible character ψ of A.

Suppose $n = 2m$. Since $\theta_1 + \theta_{m-1} = \pi$, we have $\cos\theta_1 = -\cos\theta_{m-1}$ and $\sin\theta_1 = \sin\theta_{m-1}$. Since $(\phi_1, \phi_1) + (\phi_{m-1}, \phi_{m-1}) \in \mathbf{Q}$ it follows from (22) that

$$\left[\frac{(s-1)^2}{s} + \frac{(t-1)^2}{t}\right] \frac{m}{\sin^2\theta_1} \in \mathbf{Q}.$$

If $s = 1 = t$ then (P, L, F) is an ordinary polygon. If either s or t is different from 1 we conclude that $\sin^2\theta_1 = \sin^2(\pi/m) \in \mathbf{Q}$. Thus the field of m-th roots of unity is at most quadratic over $\mathbf{Q}$ so $m \in \{1, 2, 3, 4, 6\}$ and $n \in \{2, 4, 6, 8, 12\}$. Since $(\phi_j, \phi_j) - (\phi_{m-j}, \phi_{m-j}) \in \mathbf{Q}$ for all $j = 1,..., m-1$ we conclude again using (22) that

$$\frac{(s-1)(t-1)}{s_0 t_0} \frac{2m\cos\theta_j}{\sin^2\theta_j} \in \mathbf{Q}.$$

If both s and t are different from 1 this means $(s_0t_0)^{-1}\cos\theta_j \in \mathbf{Q}$ for all $j = 1,..., m-1$. This excludes $n = 12$ and forces the restrictions $s_0t_0 \in \mathbf{Q}$ if $n = 6$ and $\sqrt{2}\, s_0t_0 \in \mathbf{Q}$ if $n = 8$.

Suppose $n = 2m + 1$. If $s > 1$ we conclude from (23) taking $j = 1$ that $\cos(2\pi/2m + 1) \in \mathbf{Q}$. Thus the field of n-th roots of unity is at most quadratic over $\mathbf{Q}$, so $n = 3$. This completes the proof.

We may also conclude from the orthogonality relations that

$$1 = (\phi_V, \mathrm{ind}) = \frac{1}{|F|} \sum_{\gamma\in\Gamma} (\mathrm{ind}\,\gamma)^{-1}(\mathrm{ind}\,\gamma)^2$$

so that we have

$$\begin{aligned} |F| &= \sum_{\gamma\in\Gamma} \mathrm{ind}\,\gamma \\ &= \begin{cases} (1+s)(1+t)(1+st+\cdots+s^{m-1}t^{m-1}), & \text{if } n = 2m, \\ (1+s)(1+s+\cdots+s^{2m}), & \text{if } n = 2m+1. \end{cases} \end{aligned} \tag{24}$$

Since $(t+1)\,|P| = |F| = (s+1)\,|L|$ this also gives formulas [4, §7.3] for $|P|$ and $|L|$.

5. Concluding Remarks

Let $V_{\mathbf{Q}} = \bigoplus_{x \in F} \mathbf{Q}x$ be the rational vector space with F as basis. We may view each $\gamma \in \Gamma$ as an endomorphism of $V_{\mathbf{Q}}$. Let $A_{\mathbf{Q}} = \bigoplus_{\gamma \in \Gamma} \mathbf{Q}\gamma$ so that $A_{\mathbf{Q}} \otimes \mathbf{C} \simeq A$. If $n \in \{2, 3, 4, 6, 8\}$ than the numbers a_j, b_j in the proof of Theorem 2 may be chosen rational, assuming that $2st$ is a square in case $n = 8$. Thus $A_{\mathbf{Q}}$ is split by $\mathbf{Q}$. The algebra $A_{\mathbf{Q}}$ is not split by $\mathbf{Q}$ if $n = 12$.

Let G be a finite group with (B, N)-pair and let W be the associated Weyl group. The generalized n-gon associated with G [5] satisfies $s > 1$ and $t > 1$, so n is not 12 and thus $A_{\mathbf{Q}}$ is split by $\mathbf{Q}$. Since G permutes the set of flags we may view $V_{\mathbf{Q}}$ as $\mathbf{Q}[G]$-module. Let $C_{\mathbf{Q}}$ be the algebra of endomorphisms of $V_{\mathbf{Q}}$ generated by G acting as group of linear transformations of $V_{\mathbf{Q}}$. From the definition of σ and τ we see that they centralize $C_{\mathbf{Q}}$ and thus the algebra $A_{\mathbf{Q}}$ they generate is included in the centralizer algebra of $C_{\mathbf{Q}}$. This centralizer algebra is called the Hecke algebra of G with respect to B and is denoted $H_{\mathbf{Q}}(G, B)$ in [6]. Thus $A_{\mathbf{Q}} \subseteq H_{\mathbf{Q}}(G, B)$. From [6] we know that $\dim H_{\mathbf{Q}}(G, B) = |W| = 2n$. Since $\dim A_{\mathbf{Q}} = 2n$ we have $A_{\mathbf{Q}} = H_{\mathbf{Q}}(G, B)$. Since $A_{\mathbf{Q}}$ is split by $\mathbf{Q}$, double centralizer theory shows that $C_{\mathbf{Q}}$ is also split by $\mathbf{Q}$. Thus if M is any simple $\mathbf{Q}[G]$-submodule of $V_{\mathbf{Q}}$, $M \otimes \mathbf{C}$ is a simple $\mathbf{C}[G]$-module. Another way to put this is to say that every absolutely irreducible character of G which appears in the induced character $1_B{}^G$ is the character of a rational representation of G. The centralizer theory also tells us that there is a bijective correspondence between simple $A_{\mathbf{Q}}$-modules M and simple $C_{\mathbf{Q}}$-modules N such that the multiplicity of M as submodule of $V_{\mathbf{Q}}$ is equal to the dimension of N. Thus the multiplicities which we have computed in Section 3 of this paper are the degrees of certain irreducible representations of G. For the case of Chevalley groups these degrees have been computed in [2].

The situation in case $n = 8$ is peculiar. The Weyl group W is dihedral of order 16. We know from Iwahori-Tits that $H_{\mathbf{C}}(G, B) \simeq \mathbf{C}[W]$. On the other hand the characters of W are not rational so $\mathbf{Q}$ does not split $\mathbf{Q}[W]$ and thus the algebras $H_{\mathbf{Q}}(G, B)$ and $\mathbf{Q}[W]$ are not isomorphic.

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